STOCHASTIC CONTROL: ALTERNATIVE TOOL IN INSURANCE RISK MANAGEMENT

M. Castillo

School of Actuarial Studies, University of New South Wales, Australia

Abstract. This survey paper aims to present recent trends and development in insurance risk management. In particular, we look at the methods and application of sophisticated risk control in managing an insurance business. As illustration, an investment portfolio of an insurance business is considered and the optimal investment strategy is determined to satisfy conditions on solvency. The investment strategies are determined using stochastic control methods.

Key-words: Insurance risk, stochastic control, dynamic programming

1 Introduction

One of the mathematically more exciting areas in actuarial mathematics is collective risk theory for non-life insurance. Here, one is concerned with the dynamics of the risk process often modeled as

$$U(t) = u + pt - \sum_{k=1}^{N(t)} X_k$$

where $u \geq 0$ is the initial surplus, $p > 0$ is the premium rate, $\{N(t), t \geq 0\}$ is a Poisson process with intensity λ, and $\{X_k : k = 1, 2, \ldots N(t)\}$ is a family of independent and identically distributed positive random variables which are independent of $\{N(t), t \geq 0\}$. This classical risk model was first considered by Filip Lundberg in 1903. Of major interest is the occurrence of ruin which is defined to be the event when surplus becomes negative. Here, we investigate infinite time ruin probability defined as

$$\psi(u) = \Pr\{U(t) < 0 \text{ for some } t \geq 0\}. \tag{1}$$

This probability is a commonly used solvency measure for an insurance business. One would like to minimize this probability subject to the dynamics of the insurance surplus process.

2 Stochastic control

Recently, a lot of interest is generated by the use of mathematical tools from stochastic control theory in addressing the problem of minimizing the infinite time ruin probability defined in (1). It has been noted that many control variables such as reinsurance, dividend payment or investment are adjusted dynamically. By means of a standard control tool such as the Hamilton-Jacobi-Bellman equation, optimal solutions can be characterized and computed, often numerically, and the smoothness of the value can be shown. Stochastic control theory methods and applications to finance are covered in the books by Yong [22], Fleming and Soner [6], and by Karatzas and Shreve [17]

Consider an insurance company managing the risk in a portfolio. Hipp [8] identified a collection of possible business strategies that may be taken by the company for risk management and discussed how the optimal strategy, which is subject to control variables and business objectives, is determined. Possible control variables include reinsurance, investment, volume control, portfolio selection, and combinations of all these actions. The optimal strategy is determined dynamically, selected and changed at each point in time depending on the risk position of the business.

The problem considered here involves finding the best strategy to satisfy particular objectives of the insurance business. Once the control variable is selected, the stochastic control problem is defined. The control problem is then solved via the Hamilton-Jacobi-Bellman (HJB) equation. The solution determines the optimal business strategy for the company. Hipp outlined the procedure for defining and solving the optimal control problem as follows:

1. Write down the controlled risk process for a constant control and its infinitesimal generator.

2. Write down the HJB equation for the stochastic control problem.

3. Show that the equation has a smooth solution satisfying the natural boundary conditions of the optimization problem.

4. Use the verification argument to show that the solution of the HJB equation is the value function of the optimization problem, and the maximizer in the equation determines the optimal strategy in feedback form.

3 Minimizing ruin with constant amount of capital for investment: an illustration

In this section, we will consider the problem of finding the optimal investment strategy that would minimize infinite time ruin probability for a particular insurance business. Details of proofs are found in Castillo and Parrocha [5]. Examples are also found in the same paper.

Dynamics of the business surplus

We model the surplus process of an insurance business whose risk process $\{R(t), t \geq 0\}$ follows a Cramér-Lundberg process

$$dR(t) = cdt - dS(t),$$

with c as the loaded premium rate and $\{S(t), t \geq 0\}$ the random claims process consisting of a sum of independent, identically distributed claims X_i, having the same distribution as X. That is,

$$S(t) = \sum_{i=1}^{N(t)} X_i$$

and $N(t)$ stands for the number of claims until time t and is modelled by a homogeneous Poisson process with constant intensity λ.

We consider a scenario where an insurance business is near ruin and the surplus or any part of it is not available for investment. A fixed amount A, independent of the current business surplus, is available for investment. The investment portfolio consists of a non-risky asset whose value $B(t)$ grows like a savings account with growth rate ρ, i.e.

$$dB(t) = \rho B(t)dt, \qquad \rho \geq 0,$$

and a risky asset whose value $Z(t)$ is modeled by a geometric Brownian motion

$$dZ(t) = \mu Z(t)dt + \sigma Z(t)dW(t), \qquad \mu \geq 0, \sigma > 0$$

where $\{W(t), t \geq 0\}$ is a standard Wiener process.

More precisely, the insurance business has the following investment policy:

- A fixed amount A, independent of the business surplus, will be invested at time t.
- A fraction $b(t) \in [0,1]$ of A will be invested at time t in the risky asset, the remaining part in the non-risky asset.
- The fraction $b(t)$ may change through time depending on which combination of risky and non-risky asset minimizes the infinite time ruin probability.

The investment return process $\{I(t), t \geq 0\}$ from the amount A is defined by

$$dI(t) = A[1-b(t)]\rho dt + Ab(t)\mu dt + Ab(t)\sigma dW(t), \tag{2}$$

and the surplus process $\{U(t), t \geq 0\}$ for this business is then seen to be

$$dU(t) = cdt - dS(t) + A[1-b(t)]\rho dt + Ab(t)\mu dt + Ab(t)\sigma dW(t). \tag{3}$$

Clearly, $U(t)$ depends on the composition of the investment portfolio in which the fixed amount A is invested and is thus influenced by the investment strategy $b(t)$.

The dynamics of the surplus process on the interval $[t, t+dt]$ can be further described as follows:

- A claim of amount X occurs with probability $\lambda dt + o(dt)$.

- No claim occurs with probability $1-\lambda dt + o(dt)$.
- An amount $cdt + o(dt)$ is received as a premium income.
- An amount $A[1-b(t)]\rho dt + o(dt)$ is received as an investment income from the non-risky asset.
- An amount $Ab(t)\mu dt + Ab(t)\sigma dW(t) + o(dt)$ is received as an investment income from the risky asset.

The control problem

Our initial aim here is to minimize the infinite time ruin probability over all possible strategies $b(t)$. The control problem can be stated as follows:

$$\underset{b(t)\in[0,1]}{\text{minimize}} \quad Pr\{U(t)<0 \quad \text{for} \quad t\geq 0\}$$

subject to

$$dU(t) = dR(t) + dI(t), \quad t \geq 0$$
$$U(0) = u_0.$$

The solution to the control problem is found via the HJB equation of the control problem. The solution to this equation determines the optimal proportion, $b_*(t)$, which is a function of the business surplus at time t. The optimal investment strategy is defined via the feedback equation

$$b_*(t) = b_*(U(t))$$

where $U(t)$ is the surplus at time t resulting from the investment strategy $\{b_*(s), s<t\}$.

The Hamilton-Jacobi-Bellman equation

We define the probability of non-ruin in the infinite time horizon, also known as the probability of survival, for a business with current surplus u to be

$$\delta(u) =: 1-\psi(u).$$

Here, we consider two distinct cases over the time interval $[t, t+dt]$

- there is no claim during the period and the surplus of the business grows by $cdt + dI(t)$, where $dI(t)$ is given in (2); and
- there is exactly one claim during the period and the surplus of the company reduces by $dI(t) - X$, where X is the random claim size. If there is a claim during the period $[t, t+dt]$, we assume that no premium is received for that period.

For an arbitrary strategy $b(t)$, at current surplus level u, the probability of survival $\delta(u)$ is determined. Taking expectations over the interval $[t, t+dt]$,

$$\delta(u) = \lambda dtE[\delta(u-X)] + (1-\lambda dt)\delta(u + cdt + dI(t)).$$

Applying Ito's lemma,

$$\delta(u) = \delta(u) + \left\{ \begin{matrix} \frac{1}{2}\sigma^2 A^2 b^2 \delta''(u) + \{c + A[1-b]\rho + Ab\mu\}\delta'(u) \\ +\lambda E\left[\delta(u-X) - \delta(u)\right] \end{matrix} \right\} dt \quad (4)$$

is obtained where the proportion invested in the risky asset depends only on the current surplus level. For convenience, we denoted this proportion simply as b keeping in mind that this changes dynamically depending on the current surplus level u. Letting $\delta(u)$ be the value function for this control problem i.e. the supremum over all possible strategies b, equation (4) then leads to

$$0 = \sup_{b\in[0,1]} \left\{ \begin{matrix} \frac{1}{2}\sigma^2 A^2 b^2 \delta''(u) + \{c + A[1-b]\rho + Ab\mu\}\delta'(u) \\ +\lambda E\left[\delta(u-X) - \delta(u)\right] \end{matrix} \right\}, \quad (5)$$

where we have the natural conditions $\delta'(u) \geq 0, \delta''(u) \leq 0$ for $u > 0, \delta(u) = 0$ for $u < 0$ and $\lim_{u\to\infty} \delta(u) = 1$. Furthermore, we assume that $\delta(u)$ is continuous on $[0,\infty)$ and twice continuously differentiable on $(0,\infty)$. Equation (5) is the HJB equation of the control problem.

The optimal investment strategy

Since $\delta''(u) \leq 0,$ the quantity inside the braces in (5) is maximized by

$$\tilde{b} = -\frac{(\mu-\rho)\delta'(u)}{A\sigma^2\delta''(u)} \quad (6)$$

and is seen to be a function of the current surplus only. Substituting (6) in (5) results in the differential equation

$$\lambda E\left[\delta(u-X) - \delta(u)\right] = \frac{1}{2}\frac{(\rho-\mu)^2\left[\delta'(u)\right]^2}{\sigma^2\delta''(u)} - (c + A\mu)\delta'(u).$$

The solution $\delta(u)$ to this equation determines the proportion required when the current surplus is u.

The verification theorem

The following theorem verifies that $\tilde{b} \in [0,1]$ is the optimal proportion when the current surplus is u. Details of the proof are in Castillo and Parrocha [5].

Theorem 1. Suppose there exists a solution $\delta_{\tilde{b}}(u)$ to the HJB equation (5), having maximizer defined in (6) such that $\delta_{\tilde{b}}(0)>0,\ \delta_{\tilde{b}}^{'}(0)>0,\ \delta_{\tilde{b}}(u)=0$ for $u<0$, $\lim_{u\to\infty}\delta_{\tilde{b}}(u)=1$, and $\delta_{\tilde{b}}(u)$ is twice continuously differentiable on $\{u>0\}$. Then if $b(t)$ is an arbitrary admissible investment strategy, for which the corresponding surplus process $\{U_b(t),t\geq 0\}$ is defined on $0\leq t<\infty$, then the corresponding non-ruin probability $\delta_b(u)$ for this process with initial surplus u satisfies

$$\delta_b(u)\leq\delta_{\tilde{b}}(u),\qquad u\geq 0.$$

Note that $\tilde{b}$, as given in (6), is not necessarily in the interval $[0,1]$. For values of $\tilde{b}$ outside the interval, it will be necessary to consider the dynamics of the business at the interval endpoints. Observe that if $\tilde{b}$ is less than 0, then $\rho>\mu$ and the non-risky asset is more advantageous than the risky asset. In this case, we do not invest in the risky asset. A different scenario is achieved when $\tilde{b}$ is greater than 1. This time, the risky asset is more advantageous than the non-risky asset and the optimal strategy therefore is to invest the full amount A on the risky asset. Since (5) is quadratic in b, the supremum value of the quantity inside the braces is therefore attained when $b=0$, $b=1$, or $b=\tilde{b}$. More specifically, the optimal strategy $b_*(t)$ at when the surplus is $U(t)$ is determined as follows:

$$b_*(t)=\begin{cases}0 & \text{if} \quad \tilde{b}<0\\ -\dfrac{(\mu-\rho)\delta^{'}(U(t))}{A\sigma^2\delta^{''}(U(t))} & \text{if} \quad 0\leq\tilde{b}\leq 1\\ 1 & \text{if} \quad \tilde{b}>1.\end{cases} \tag{7}$$

As a special case, if the current surplus is zero we do not invest in the risky asset. This is evident from the fact that if the current surplus is 0 and most of the amount A is invested in the risky asset then there will be greater chance of shortage of surplus to pay out possible early claims. It will follow from (5) that

$$\begin{aligned}0&=(c+A\rho)\delta^{'}(0)+\lambda E\left[\delta(0-X)-\delta(0)\right]\\ &=(c+A\rho)\delta^{'}(0)-\lambda\delta(0),\end{aligned}$$

and

$$\delta^{'}(0)=\frac{\lambda\delta(0)}{c+A\rho}. \tag{8}$$

The optimal non-ruin probabilities

The non-ruin probabilities $\delta(u)$ for cases $b_*(t)=0$, $b_*(t)=1$, and $b_*(t)=\tilde{b}$

are characterized here. Details are in Castillo and Parrocha [5]. Properties of these probabilities, denoted by $\delta_0(u)$, $\delta_1(u)$, and $\delta_{\tilde{b}}(u)$, respectively, are used in proving the existence of a solution to the HJB equation.

Case 1: $b_*(t) = 0$

$$\delta_0'(u) = \frac{\lambda}{c + A\rho} E\left[\delta_0(u) - \delta_0(u - X)\right]. \tag{9}$$

Case 2: $b_*(t) = 1$

$$\delta_1'(u) = \frac{2}{\sigma^2 A^2} \int_{u_1}^{u} \left\{\lambda E[\delta_1(t) - \delta_1(t - X)] - (c + A\mu)\delta_1'(t)\right\} dt + \delta_1'(u_1). \tag{10}$$

Case 3: $b_*(t) = \tilde{b}$

$$\delta_{\tilde{b}}'(u) = \left\{ \begin{array}{l} \dfrac{(\rho - \mu)^2}{2\sigma^2} \displaystyle\int_0^u \frac{1}{\lambda E[\delta_{\tilde{b}}(t) - \delta_{\tilde{b}}(t - X)] - (c + A\rho)\delta_{\tilde{b}}'(t)} dt \\ + \dfrac{c + A\rho}{\lambda \delta_{\tilde{b}}(0)} \end{array} \right\}^{-1}. \tag{11}$$

Existence of a solution to the HJB equation

Equation (5) determines solutions up to a multiplicative constant, i.e. if $\delta(u)$ is a solution then it follows that $g(u) = \omega\delta(u)$, where $\omega > 0$ solves (5) with boundary condition $g(\infty) = \omega$. The proof considers a solution using $g(0) = \delta_0(0)$.

Using the function $g(u)$ instead of $\delta(u)$, (11) can be transformed to

$$g_{\tilde{b}}'(u) = \left\{ \begin{array}{l} \dfrac{(\rho - \mu)^2}{2\sigma^2} \displaystyle\int_0^u \frac{1}{\lambda E[g_{\tilde{b}}(t) - g_{\tilde{b}}(t - X)] - (c + A\rho)g_{\tilde{b}}'(t)} dt \\ + \dfrac{c + A\rho}{\lambda g_{\tilde{b}}(0)} \end{array} \right\}^{-1}. \tag{12}$$

We first show that the integral in (12) is finite for a surplus u. The procedure starts by showing that $g_{\tilde{b}}(u)$ exists on an interval close to zero.

Express $E[g(t) - g(t - X)]$ in terms of $g'(t)$. By the definition of the expectation with $F(x)$ and $f(x)$, the distribution and density function of X respectively,

$$\begin{aligned} E[g(t)-g(t-X)] &= g(t)-\int_0^t g(t-x)f(x)dx \\ &= g(t)-g(0)F(t)+g(t)F(0)-\int_0^t F(x)g'(t-x)dx \\ &= g(t)-g(0)F(t)-\int_0^t F(t-z)g'(z)dz \\ &= g(0)[1-F(t)]+\int_0^t \left[1-F(t-z)\right]g'(z)dz. \end{aligned}$$

If the expression $E[g_{\tilde{b}}-g_{\tilde{b}}(t-X)]$ in (12) is replaced by a corresponding expression based on the formula above, the equation becomes

$$g_{\tilde{b}}'(u)=\left[\begin{array}{l} \dfrac{(\rho-\mu)^2}{2\sigma^2}\displaystyle\int_0^u \dfrac{dt}{\left\{\begin{array}{l}\lambda\left\{g_{\tilde{b}}(0)\left[1-F(t)\right]+\int_0^t\left[1-F(t-z)\right]g_{\tilde{b}}'(z)dz\right\} \\ -(c+A\rho)g_{\tilde{b}}'(t)\end{array}\right\}} \\ +\dfrac{c+A\rho}{\lambda g_{\tilde{b}}(0)}\end{array}\right]^{-1}.$$

Define a function $k(u)$ by

$$k(u)=\frac{\frac{\lambda g_{\tilde{b}}(0)}{c+A\rho}-g_{\tilde{b}}'(u^2)}{u}. \tag{13}$$

Then we have

$$\begin{aligned} &\tfrac{\lambda g_{\tilde{b}}(0)}{c+A\rho}-uk(u) \\ &=\left[\begin{array}{l} \dfrac{(\rho-\mu)^2}{\sigma^2}\displaystyle\int_0^u \dfrac{dt}{\left\{\begin{array}{l}(c+A\rho)k(t)-\lambda g_{\tilde{b}}(0)\frac{F(t^2)}{t}+ \\ \lambda\int_0^1\left[1-F(t^2-t^2z)\right]\left[\frac{\lambda g_{\tilde{b}}(0)}{c+A\rho}-t\sqrt{z}k(t\sqrt{z})\right]tdz\end{array}\right\}} \\ +\dfrac{c+A\rho}{\lambda g_{\tilde{b}}(0)}\end{array}\right]^{-1} \end{aligned}$$

Solving for $k(u)$,

$$k(u)=\frac{l(u)}{u}\frac{\lambda^2 g_{\tilde{b}}(0)^2(\rho-\mu)^2}{\lambda g_{\tilde{b}}(0)(c+A\rho)(\rho-\mu)^2 l(u)+\sigma^2(c+A\rho)^2} \tag{14}$$

where the function $l(u)$ is defined by

$$l(u)=\int_0^u \frac{dt}{\left\{\begin{array}{l}(c+A\rho)k(t)-\lambda g_{\tilde{b}}(0)\frac{F(t^2)}{t} \\ +\lambda t\int_0^1\left[1-F(t^2-t^2z)\right]\left[\frac{\lambda g_{\tilde{b}}(0)}{c+A\rho}-t\sqrt{z}k(t\sqrt{z})\right]dz\end{array}\right\}}.$$

Note that $\lim_{t\to 0}\frac{F(t^2)}{t}=\lim_{t\to 0}2tf(t^2)=0.$ Furthermore, the functions $F(x)$ and $k(u)$ present in the integrand defining the function $l(u)$ are bounded. Therefore, the inner integral and the integrand itself are bounded and the following statements hold:

- $\lim_{u\to 0} l(u)=0$
- $\lim_{u\to 0}\frac{l(u)}{u}=\frac{1}{(c+A\rho)\lim_{u\to 0}k(u)}$

Taking the limit of both sides of (14) as $u\to 0$ and applying the preceding properties, we have

$$\left[\lim_{u\to 0}k(u)\right]^2=\frac{\lambda^2 g_{\tilde{b}}(0)^2(\rho-\mu)^2}{\sigma^2(c+A\rho)^3}.$$

Equivalently,

$$\lim_{u\to 0}k(u)=-\frac{\lambda g_{\tilde{b}}(0)(\rho-\mu)}{\sigma(c+A\rho)^{\frac{3}{2}}}.$$

Now that we know the behavior of $k(u)$ for small values of u, it is possible to generate a corresponding behavior for $g_{\tilde{b}}'(u)$. Using (13),

$$g_{\tilde{b}}'(u)=\frac{\lambda g_{\tilde{b}}(0)}{c+A\rho}-k(\sqrt{u})\sqrt{u}.$$

As $u\to 0$ it follows that

$$g_{\tilde{b}}'(u)=\frac{\lambda g_{\tilde{b}}(0)}{c+A\rho}+\frac{\lambda g_{\tilde{b}}(0)(\rho-\mu)}{\sigma(c+A\rho)^{\frac{3}{2}}}\sqrt{u}. \tag{15}$$

Equation (15) gives the derivative of $g_{\tilde{b}}(u)$ for small values of u. Since $g_{\tilde{b}}(0)$ is known, the above equation can be integrated to get $g_{\tilde{b}}(u)$. Therefore, a solution to (12) exists.

4 Constraints and limitations

Finding explicit solutions to the Hamilton-Jacobi-Bellman equation is never possible in the framework considered here. However, the existence proof renders a good numerical method for computations.

5 Review of past results

Early papers on stochastic control in insurance include Martin-Lof, Brockett and Xia, and Browne [3]. Recent applications of stochastic control tools in minimizing the probability of ruin include the optimization of reinsurance programs (see Hipp and Vogt [13], Hojgaard and Taksar [14], [15], Taksar and Markussen [21] and Schmidli [20]), the issuance of new business (see Hipp and Taksar [12]), optimal investment strategies (see Hipp and Plum [10], Hipp and Schmidli [11], and Gaier et.al [7]), and simultaneous dynamic control of reinsurance and investment (see Schmidli [24]).

Other results that have as objective the maximization of cumulative expected discounted dividend payouts for various control variables include the papers written by Hipp and Plum [9], Asmussen et. al [1], Hojgaard and Taksar [16] and Paulsen [19].

6 Suggestions for further work

Most of the papers mentioned above assume the classical compound Poisson process in the claims arrival process of an insurance company. from a practical point of view, there is a need for a claim arrival process that allow for jumps. As an alternative process to generate claims, we can employ the Cox process or a doubly stochastic Poisson process. The paper by Hipp and Plum [9] considers an optimal control problem where a stochastic discount rate is considered, the stochastic model is a time homogeneous finite state Markov process. This can be extended to a scenario where discount rates are generated through a Cox process or a doubly stochastic Poisson process. Allowing for jumps in the risk process clearly present new theoretical and computational issues but would hopefully provide new insight and more realistic insurance risk models.

References

[1] Browne, S. (2000) *Optimal risk control and dividend distribution policies. example of excess-of loss reinsurance for an insurance corporation.* Finance Stochastics, 4, 299-324.

[2] Brockett, P. and Xia, X. (1995) *Operations research in insurance: a review.* Trans. Act. Soc., 47, 7-80.

[3] Browne, S. (1995) *Optimal investment policies for a firm with a random risk process: exponential utility and minimizing the probability of ruin.* Mathematics of Operations Research, 20, 937-958.

[4] Buhlmann, H. (1996) Mathematical methods in risk theory. Berlin Heidelberg: Springer-Verlag.

[5] Castillo, M. and Parrocha, G. (2004) *Stochastic Control Theory for Optimal Investment.* ARCH2004.1. Download available from http://library.soa.org/library-pdf/arch04v38n1_17.pdf

[6] Fleming, W.H. and Soner, M. (1993) Controlled Markov processes and viscosity solutions. New Yor: Springer.

[7] Gaier, J., Grandits, P. and Schachermayer (2003) *Asymptotic ruin probabilities and optimal investments.* Ann. Appl. Probab. 13, no. 3, 1054–1076.

[8] Back, K. et al. (2004) *Stochastic Methods in Finance.* Lecture Notes in

Mathematics 1856, pp.127-2004. Berlin Heidelberg: Springer-Verlag.

[9] Hipp, C. and Plum, M. (2003) *Optimal investment for investors with state dependent income, and for insurers.* Finance and Stochastics 7, 299-321.

[10] Hipp, C. and Plum, M. (2000) *Optimal investment for insurers.* Insurance: Mathematics and Economics 27, 251-262.

[11] Hipp, C. and Schmidli, H. (2003) *Asymptotics of the ruin probability for the controlled risk process: the small claims case.* Download available from http://insurance.fbv.uni-karlsruhe.de/download/hippschmidli.pdf.

[12] Hipp, C. and Taksar, M. (2000) *Stochastic control for optimal new business.* Insurance: Mathematics and Economics 26, 185-192.

[13] Hipp, C. and Vogt, M. (2003) *Optimal dynamic XL-reinsurance.* Download available from http://insurance.fbv.uni-karlsruhe.de/download/vogt.pdf

[14] Hojgaard, B. and Taksar, M. (1998) *Optimal proportional reinsurance policies for diffusion models.* Scandinavian Actuarial Journal, 81, 166-180.

[15] Hojgaard, B. and Taksar, M. (1999) *Optimal proportional reinsurance policies for diffusion models with transaction costs.* Insurance: Mathematics and Economics 22, 41-51.

[16] Hojgaard, B. and Taksar, M. (2001) *Optimal risk control for a large corporation in the presence of returns on investments.* Finance and Stochastics, 5, 527-547.

[17] Karatzas, I. and Shreve, S. (1997) Methods of Mathematical finance. Heidelberg: Springer.

[18] Martin-Lof, A. (1994) *Lectures on the use of control theory in insurance.* Scandinavian Actuarial Journal, 1, 1-25.

[19] Paulsen, J. (2003) *Optimal dividend payouts for diffusions with solvency constraints.* Finance and Stochastics, 7, 4, 457-474.

[20] Schmidli, H. (1999) *Optimal proportional reinsurance policies in a dynamic setting.* Scandinavian Actuarial Journal, 77, 1-25.

[21] Taksar, M. and Markussen, C. (2003) *Optimal dynamic reinsurance policies for large insurance portfolios.* Finance Stochastics, 7, 97-121.

[22] Yong, J. and Zhou, X. (1999) Stochastic controls: hamiltonian systems and HJB equations. New York: Springer-Verlag.

[23] Schmidli, H. (2001). *Optimal proportional reinsurance policies in a dynamic setting.* Scandinavian Actuarial Journal, 55-68

[24] Schmidli, H. (2002). On minimising the ruin probability by investment and reinsurance. Ann. Appl. Probab. 12, 890 -907.

MARTINA CASTILLO: Actuarial Studies, University of New South Wales,
Sydney 2052, Australia.
Phone: +61-2-93852647
E-mail: tina.castillo@unsw.edu.au