LIFE INSURANCE WITH STOCHASTIC INTEREST RATE

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Abstract. Pricing of insurance product is usually evaluated on a basis where interest rate is assumed to be fixed over time. To obtain a more realistic assessment of the pricing of its product it would be benefit if the interest rates are fluctuating. This paper compares actuarial quantities which calculated based on fixed interest rate to stochastic interest rate using the Vasicek and the Cox, Ingersoll and Ross financial valuation models. In this case, time to maturity in financial valuation models is adjusted with *T(x),* a continuous random variable representing future lifetime of a life-aged-*x.* *T(x)* is obtained through simulation based on Gompertz Mortality Law. By means of Monte Carlo simulation we calculate actuarial quantity under whole life insurance and give empirical results. Furthermore, we quantify Wang's Transform risk measure with respect of loss distribution.

Key-words: *life insurance pricing, interest rate derivatives, Gompertz mortality law, Monte Carlo simulation.*

1 Introduction

During the last year, insurance pricing models involving stochastic interest rate have become more and more interesting. It is worthwhile to understand that one has to be careful when working with stochastic interest rate. New environment, not known in a world of deterministic interest rate especially when combining with finance which have used more sophisticated mathematical concepts, such as martingales or stochastic integration in order to describe the economic behavior or to derive computing methods such as the absence of arbitrage opportunity and equilibrium theory.. Enormous literature in finance such as Vasicek (1977) and Cox, Ingersoll and Ross (CIR) (1985) have documented that interest rate should be followed by a stochastic process.

The remainder of the paper is organized as follows. Section 2 is started with mortality as the basic building block in actuarial science. In section 3, we determine the actuarial quantities based on interest rate derivatives models. In section, 4 we simulate and give empirical results. Section 5 is closed with a conclusion.

2 Mortality

Symbol *(x)* will be used to denote a *life-aged-x* (Bowers et.al. 1997). Mortality can be stated by *K(x)* discrete random variable representing the number of completed future years lived by *(x)* or *T(x)* continuous random variable representing future lifetime of *(x).* The mortality table describes not only completely *K(x)* distribution but also can describe *T(x)* distribution through the approximation.

The three characteristics of Gompertz Mortality Law are as follows

- The force of mortality;

$$\mu_{x+t} = BC^{x+t} \qquad B > 0,\ C > 0,\ t > 0$$

- The density function;

$$f(t)\ dt = \left(-BC^{x+t}\right)\exp\left(-\frac{BC^x}{\ln C}\left(C^t - 1\right)\right)$$

- The distribution function;

$$\begin{aligned} F(t) &= P(T(x) \le t) \qquad\qquad t \ge 0 \\ &= 1 - \mathrm{e}^{-\frac{BC^x}{\log C}\left(C^t - 1\right)} \\ &= 1 - e^{M(t)} \end{aligned} \tag{1}$$

Parameter *B* and *C* are estimated by nonlinear least square based on the U.S. Mortality Table 1979-1981 (Noviyanti, L. and Syamsuddin, M., 2003).

It is known that *M(t)* random variable in equation (1) has exponential distribution (λ=*1*). To simulate *M(t),* let *U* be a uniform *(0,1)* random variable. According to Ross (1977), *M* = - *ln(U)* constitutes random variable of exponential distribution (λ = *1*).

T(x) can be simulated using the inversion method (Pai, 1997) based on *M(t)* random variable. It will be obtained *T* random variable of Gompertz Mortality Law

$$T = \ln\left(\tfrac{M\ \ln(C)}{BC^x} + 1\right) / \ln(C)$$

3 Actuarial Quantity

The present value of a benefit at time *t* is defined by

$$z_t = b_t\, v_t$$

where b_t is the amount of benefit at time *t* and v_t is the discount factor. Calculating a present value usually assumes that its value depends on the length of the time period and the force of interest / the interest rate is constant. We denote Z_t as the random variable with outcome z_t. The expectation of the present value random variable, $E(Z_t)$, is called the
actuarial present value for the whole life insurance with a unit payable at the moment of death of (x), denoted by $\bar{A}_x$.

$$\bar{A}_x = E\left(v^t\right) \tag{2}$$
$$= \int_0^\infty v^t \, f_x(t)\, dt$$

The actuarial present value for a continuous whole life annuity is

$$\bar{a}_x = \int_0^\infty v^t \; {}_t p_x \; dt$$

Quantity the discount factor $v^t = exp\ (-\delta t)$ represents the present value at time zero of one unit of account at time *t* discounted by constant interest rate. The discount factor discounted by stochastic interest rate is

$$v(t) = \exp\left(-\int_0^t r(s)\, ds \right)$$

We recall the interest rate derivative models. Securities with payoffs that depend on interest rates are called interest rate derivatives. Such securities are important because almost every financial transaction entails exposure to interest rate risk and interest rate derivatives provide the means for controlling that risk. Interest rate derivative securities are relevant to many forms of investment and the Vasicek model and the CIR model are based on zero coupon bond.

In this approach, it is specified that the instantaneous short rate *r(t)* satisfies an equation of the Ito equation, showing the relationship between derivative price changes and the interest rate and time changes.

$$\mathrm{dr(t)} = \mu(\mathrm{r,t})\,\mathrm{dt} + \sigma(r,t)\ \mathrm{dW(t)}$$

Given an initial condition *r(0)*, the equation defines a stochastic process *r(t)* (Lamberton, D., and Lapeyre, B., 2000). Evolution of interest rates is driven by the short rate *r(t)* and short rates are reverting with a constant reversion rate.

It is assumed the interest rates follow the stochastic process suggested by Vasicek and Cox, Ingersoll and Ross (Brigo, D. and Mercurio, F., 2001) and can be expressed as follows

- The Vasicek Model

$$\mathrm{dr(t)} = \mathrm{c}(\theta\text{-}\mathrm{r(t)})\mathrm{dt} + \sigma\ \mathrm{dW(t)}$$

r(t)	: the short term interest rate
c	: the speed of adjustment in mean reverting process
θ(t)	: the long run average value of *r(t)*
σ	: the standard deviation of the interest rate process
W(t)	: a standardized Weiner process

- The Cox, Ingersoll and Ross Model

$$dr(t) = c(\theta\text{-}r(t))dt + \sigma\sqrt{r(t)}\, dW(t)$$

The process for *r(t)* involves only sources of uncertainty driving all rates. This usually means that in any short period of time all rates move in the same direction. The drift *[c(θ-r(t))]* and σ are assumed to be functions of *r*, but are independent of time.

According to the CIR model, the standard deviation of the change in the short rate in a short period of time is proportional to $r(t)^{1/2}$. This means that, as the short-term interest rate increases, its standard deviation increases. Differs from the Vasicek model, the CIR model eliminates the possibility of negative interest rates.

Let *P(t)* denotes the current of a one-dollar zero-coupon bond at period *t*. From the two models,

$$P(t) = E\left(\exp\left\{-\int_0^t r(s)\,ds\right\}\right)$$

$$= A(t)\,e^{-B(t)r}$$

where

$$A_{Vasicek}(t) = \exp\left\{\frac{(B(t)-t)\left(\theta - \frac{\sigma^2}{2c^2}\right)}{c^2} - \frac{\sigma^2 B(t)^2}{4c}\right\};\ B_{Vasicek}(t) = \frac{1-e^{-ct}}{c}$$

$$A_{CIR}(0,t) = \left(\frac{2\sqrt{c^2+2\sigma^2}\, e^{\frac{t}{2}(c+\sqrt{c^2+2\sigma^2})}}{\left(\sqrt{c^2+2\sigma^2}+c\right)\left(e^{t\sqrt{c^2+2\sigma^2}}-1\right)+2\sqrt{c^2+2\sigma^2}}\right)^{2c\mu/\sigma^2}$$

$$B_{CIR}(0,t) = \frac{2\left(e^{t\sqrt{c^2+2\sigma^2}}-1\right)}{\left(\sqrt{c^2+2\sigma^2}+c\right)\left(e^{t\sqrt{c^2+2\sigma^2}}-1\right)+2\sqrt{c^2+2\sigma^2}}$$

Furthermore, the actuarial present values for the whole life insurance with a unit payable at the moment of death of (x) and a continuous whole life annuity based on the two models of interest rate derivatives are denoted by $\bar{A}_{xp}$ and $\bar{a}_{xp}$.

$$
\begin{aligned}
\bar{A}_{xp} &= E\,(E\,(v(t)) \\
&= \int_0^\infty E\,(v(t))\; f_{Tx}(t)\,dt \qquad (3) \\
&= \int_0^\infty P\,(0,t)\; f_{Tx}(t)\,dt
\end{aligned}
$$

$$
\bar{A}_{xpVasicek} = \int_0^\infty \exp\left(-t\left(\mu - \frac{\sigma^2}{2c^2}\right) - \frac{r(t) - \left(\mu - \frac{\sigma^2}{2c^2}\right)}{c}\left(1-e^{-ct}\right) - \frac{\sigma^2}{4c^3}\left(1-e^{-ct}\right)^2 \right) f(t)\,dt
$$

$$
\bar{A}_{xpCIR} = \int_0^\infty \left(\frac{2d\,e^{t(d+c)/2}}{d-c+e^{dt}(d+c)} \right)^{2c\mu/\sigma^2} \exp\left(-r(t)\frac{2\left(e^{dt}-1\right)}{d-c+e^{dt}(d+c)} \right) f(t)\,dt\ ;\ d = \sqrt{c^2+2\sigma^2}
$$

$$
\begin{aligned}
\bar{a}_{xp} &= \int_0^\infty P(t)\ {}_t p_x\ dt \\
&= \int_0^\infty \frac{P(t)}{\mu_{x+t}}\ f_x(t)\ dt \qquad (4) \\
&= \int_0^\infty G(t)\ f_x(t)\ dt
\end{aligned}
$$

The premium is calculated based on equivalence principle.

$$
P_{xp} = \bar{A}_{xp} / \bar{a}_{xp}
$$

After having the actuarial quantities, we calculate a loss function;

$$
L = v(T) - P\,.\,a_{\overline{T|}}
$$

The risk of company loss will be occurred when the value of loss function (L) is positive; this means that the value of benefit which will be paid is greater than premium obligation received.

The last step is to quantify Wang's Transform risk measure with respect of loss distribution; For a loss variable L with distribution F, Wang (2001) defined a new risk measure for capital requirement as follows:

- For a pre-selected security level α, let $\lambda = \Phi^{-1}(\alpha)$

- Apply the Wang Transform : $F^*(l) = \Phi[\Phi^{-1}(F(l)) - \lambda]$
- Set the capital requirement to be the expected value under F^* : $WT(\alpha) = E^*(L)$

4 Empirical Results

In this section, we present empirical results and sensitivity analysis with respect to parameters of insurance contract. To calculate $\overline{A}_x$ (equation (2)), $\overline{A}_{xp}$ (equation (3)) and $\overline{a}_{xp}$ (equation (4)) each depends on $E(v^t)$, $E(v(t))$ and $E(G(t))$. By using Monte Carlo Integration Method in Matlab 6.0 which utilizes the results the Law of Large Number, the actuarial quantities are

$$E(e^{-\delta T}) \approx \frac{1}{N}\sum_{i=1}^{N} e^{-\delta T_i}$$

$$E(P(T)) \approx \frac{1}{N}\sum_{i=1}^{N} \exp\left(-\int_0^t r(s)\,ds\right) = \frac{1}{N}\sum_{i=1}^{N} P(t)$$

$$E(G(T)) \approx \frac{1}{N}\sum_{i=1}^{N} \frac{P(t)}{\mu_{x+t}}$$

where T_i constitutes i^{th} T simulation, $i = 1,2,\ldots,N$.

Table 1 and Figures 1 to 4 display pattern the actuarial quantities under whole life insurance and Wang's Transform risk measure with changes in parameters of the speed of adjustment in mean reverting prices (c), initial volatility of interest rates (σ) and the long run average value (θ) (x = 25, 35 and 45 ; $r_0 = 0.05$).

Figure 1. Actuarial Present Values (x=35)

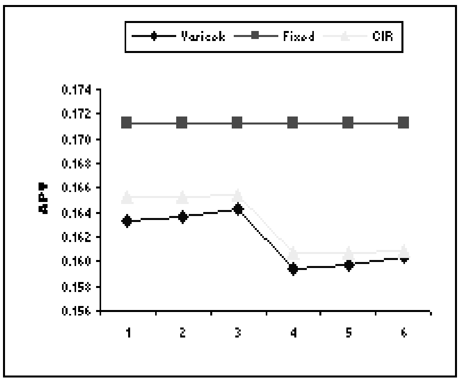


Figure 2. Annuities (x=35)

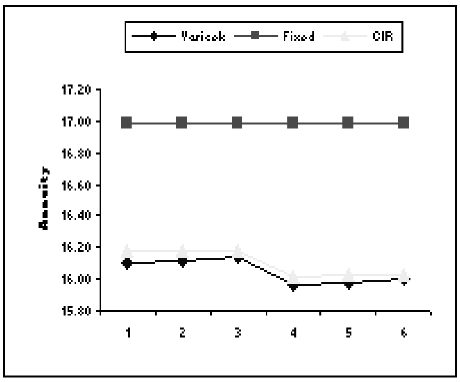


Figure 3. The Premiums (*x=35*)

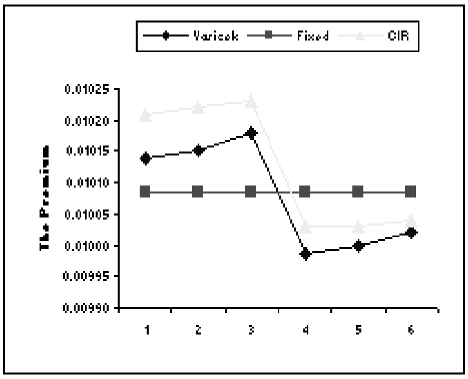
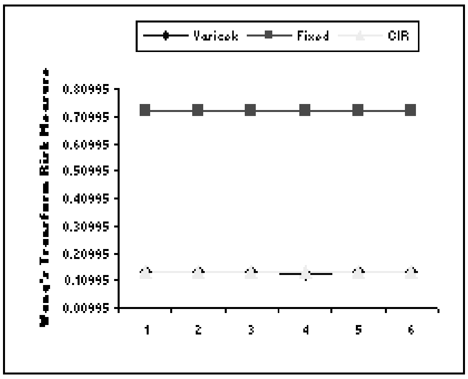


Fig. 4. Wang's Transform Risk Measures (*x=35*)



Figures 1 and 2 show that the values of APV and Annuity based on stochastic interest rates give less values than fixed interest rate although their premium values (Figure 3) are relatively close to each other ranges from *0.00999* to *0.01018* (Table 1*, x = 35*).

Furthermore the obtainable information (Figures 4) indicates that the Wang's Transform Risk Measures (*x=35* and also for *x=25* and *x=45*) of the Vasicek Model and the CIR model give also less values. It will make a great influence related to the amount of loss that will be borne by the company.

Table 1. Actuarial quantities and Wang's Transform (WT) risk measure.

x = 25 ; Tx=51.162			**APV**	**Annuity**	**Premium**	**WT**
k	**µ**	□	0.1139	18.1610	0.00627	0.6545
The Vasicek Model						
1.1	0.051	0.005	0.1458	23.0900	0.00631	0.1425
1.1	0.051	0.015	0.1461	23.1086	0.00632	0.1428
1.1	0.051	0.025	0.1469	23.1459	0.00635	0.1433
1.1	0.052	0.005	0.1412	22.8580	0.00618	0.1394
1.1	0.052	0.015	0.1416	22.8762	0.00619	0.1396
1.1	0.052	0.025	0.1423	22.9127	0.00621	0.1401
The CIR Model						
0.5	0.042	0.005	0.1453	22.9677	0.00633	0.1419
0.5	0.042	0.015	0.1454	22.9719	0.00633	0.1420
0.5	0.042	0.025	0.1456	22.9802	0.00634	0.1421
0.5	0.043	0.005	0.1400	22.7089	0.00617	0.1383
0.5	0.043	0.015	0.1401	22.7131	0.00617	0.1383
0.5	0.043	0.025	0.1403	22.7214	0.00617	0.1385

x = 35 ; Tx=41.762			**APV**	**Annuity**	**Premium**	**WT**
k	**μ**	□	0.1713	16.9850	0.01009	0.7280
The Vasicek Model						
1.1	0.061	0.005	0.1633	16.1047	0.01014	0.1395
1.1	0.061	0.015	0.1636	16.1157	0.01015	0.1397
1.1	0.061	0.025	0.1642	16.1377	0.01018	0.1401
1.1	0.062	0.005	0.1594	15.9666	0.00999	0.1371
1.1	0.062	0.015	0.1597	15.9774	0.01000	0.1373
1.1	0.062	0.025	0.1604	15.9991	0.01002	0.1377
The CIR Model						
0.5	0.05	0.005	0.1652	16.1736	0.01021	0.1408
0.5	0.05	0.015	0.1653	16.1765	0.01022	0.1408
0.5	0.05	0.025	0.1655	16.1822	0.01023	0.1409
0.5	0.051	0.005	0.1607	16.0193	0.01003	0.1379
0.5	0.051	0.015	0.1608	16.0222	0.01003	0.1379
0.5	0.051	0.025	0.1609	16.0279	0.01004	0.1381

x = 45 ; Tx=32.806			**APV**	**Annuity**	**Premium**	**WT**
k	**μ**	□	0.2505	15.3610	0.01631	0.7911
The Vasicek Model						
1.1	0.081	0.005	0.1700	10.4441	0.01628	0.1240
1.1	0.081	0.015	0.1702	10.4496	0.01629	0.1241
1.1	0.081	0.025	0.1707	10.4606	0.01632	0.1244
The CIR Model						
0.7	0.066	0.005	0.1745	10.5951	0.01647	0.1268
0.7	0.066	0.015	0.1746	10.5961	0.01648	0.1268
0.7	0.066	0.025	0.1747	10.5981	0.01648	0.1269
0.7	0.067	0.005	0.1712	10.5125	0.01629	0.1249
0.7	0.067	0.015	0.1712	10.5135	0.01629	0.1249
0.7	0.067	0.025	0.1713	10.5155	0.01629	0.1249

The simulation result (Table 1) shows, as expected, the premium values of the Vasicek Model and the CIR model are increasing in age.

Table 2 gives adjusted parameter values with respect to the close premiums between fixed and stochastic interest rates.

Table 2. The parameters of The Vasicek Model and the CIR Models
(σ = *0.005; 0.015; 0.025*)

Ages	Parameters	The Vasicek Model	The CIR Model
25	**c**	1.1	0.50 ; 0.055
	θ	0.051 ; 0.052 ; 0.053	0.042 ; 0.043
35	**c**	1.1	0.50 ; 0.55 ; 0.60
	θ	0.060 ; 0.061 ; 0.062	0.05 ; 0.051
45	**c**	1.1	0.7 ; 0.8
	θ	0.08 ; 0.0805 ; 0.081	0.066 ; 0.067

According to Table 2, it can be seen that parameter of the speed of adjustment in mean reverting process (c) of the Vasicek Model is *1.1* for all the ages, whereas parameter c of the CIR Model are varying from *0.5* to *0.8*. It shows that the CIR Model is sensitive to the change in the speed of adjustment in mean reverting process (c).

5 Conclusion

In this paper we have shown the combination of the use of financial and actuarial approaches to price life insurance contract. The Vasicek and The Cox, Ingersoll and Ross financial valuation models are used in order to calculate actuarial quantities and Wang's Transform risk measure. Simulation result shows a significantly different risk value in relation between fixed and stochastic interest rate.

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